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Applied Mathematics Letters 19 (2006) 38–44
Applied Mathematics Letters
www.elsevier.com/locate/aml

Some sufficient conditions for a graph to be of C_f 1[☆]

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Received 9 November 2004; received in revised form 23 March 2005; accepted 24 March 2005

Abstract

An f-coloring of a graph G is a coloring of edges of $E(G)$ such that each color appears at each vertex $v \in V(G)$ at most $f(v)$ times. The minimum number of colors needed to f-color G is called the f-chromatic index $\chi'_f(G)$ of G. Any graph G has f-chromatic index equal to $\Delta_f(G)$ or $\Delta_f(G)+1$, where $\Delta_f(G) = \max_{v\in V}\{\lceil \frac{d(v)}{f(v)} \rceil\}$. If $\chi'_f(G) = \Delta_f(G)$, then G is of C_f 1; otherwise G is of C_f 2. Some sufficient conditions for a graph to be of C_f 1 are given.

Keywords: Edge coloring; f-coloring; Classification of graphs

1. Introduction

Our terminology and notation will be standard. Readers are referred to [1] for undefined terms. Throughout this paper the term *graph* is used to denote a simple graph G with a finite vertex set V and a finite and nonempty edge set E. If multiple edges are allowed, G is called a *multigraph*. In the classical edge coloring, each vertex has at most one edge colored with the same color. Hakimi and Kariv [2] generalized the classical edge colorings and obtained many interesting results. The *degree* $d(v)$ of vertex v is the number of edges incident with v in the multigraph G. Let G be a multigraph and let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V$. An *f-coloring* of G is a coloring of edges such that each vertex v has at most $f(v)$ edges colored with the same color. The minimum number

☆ This research was supported by NSFC(10471078), RSDP(20040422004) and PDSF(2004036402) of China.
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0893-9659/$ - see front matter
doi:10.1016/j.aml.2005.03.006

of colors needed to f-color G is called an *f-chromatic index* of G, and denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V$, the f-coloring problem is reduced to the classical edge-coloring problem.

f-colorings have applications in scheduling problems such as the file transfer problem in a computer network. Since the classical edge-coloring problem is NP-complete [3], the f-coloring problem which asks us to f-color a given multigraph G with $\chi'_f(G)$ colors is also NP-complete in general.

Let

$$\Delta = \max_{v\in V}\{d(v)\} \qquad \text{and} \qquad \Delta_f = \max_{v\in V}\left\{\left\lceil \frac{d(v)}{f(v)} \right\rceil\right\},$$

in which $\lceil \frac{d(v)}{f(v)} \rceil$ is the smallest integer not smaller than $\frac{d(v)}{f(v)}$. It is easy to verify that $\chi'_f(G) \geq \Delta_f$. The *multiplicity* $\mu(u, v)$ of a pair of u and v of distinct vertices is the number of edges of G joining u and v. Let $\mu(v) = \max_{u\in V}\{\mu(v, u)\}$. A well-known theorem of Vizing [4] states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1,$$

where G is a graph. Graphs for which $\Delta(G) = \chi'(G)$ are said to be *class* 1, and otherwise they are *class* 2. In this paper, we generalize the classification of graphs in natural ways.

The following lemma was given by Hakimi and Kariv [2].

Lemma 1.1. *Let G be a multigraph. Then*

$$\Delta_f \leq \chi'_f(G) \leq \max_{v\in V}\left\{\left\lceil \frac{d(v)+\mu(v)}{f(v)} \right\rceil\right\}.$$

When G is a graph, we have $\mu(v) = 1$ for each $v \in V$. Therefore the following corollary holds.

Corollary 1.2. *Let G be a graph. Then*

$$\Delta_f \leq \chi'_f(G) \leq \max_{v\in V}\left\{\left\lceil \frac{d(v)+1}{f(v)} \right\rceil\right\} \leq \Delta_f + 1.$$

From the above corollary we can see that the f-chromatic index of any graph G must be Δ_f or $\Delta_f + 1$. This immediately gives us a simple way of classifying graphs into two classes according to their f-chromatic indices. More precisely, we say that G is of C_f 1 if $\chi'_f(G) = \Delta_f$; and that G is of C_f 2 if $\chi'_f(G) = \Delta_f + 1$. Hakimi and Kariv [2] obtained the following two results.

Theorem A. *Let G be a bipartite graph. Then $\chi'_f(G) = \Delta_f$.*

Theorem B. *Let G be a graph and $f(v)$ be even for all $v \in V$. Then $\chi'_f(G) = \Delta_f$.*

If $f(v) = 1$ for all $v \in V$, our classification problem on f-colorings is reduced to the classical classification problem.

This paper is organized as follows. In Section 2, we present some sufficient conditions for a graph to be of C_f 1. In Section 3, some problems for future research are given.

2. Main results

Throughout this section, G always denotes a graph.

Let

$$V^* = \left\{ v : \Delta_f = \left\lceil \frac{d(v)}{f(v)} \right\rceil, v \in V \right\}.$$

The following Theorem 2.1 is easy to verify, but it is very important for our classification problem on f-colorings.

Theorem 2.1. *Let G be a graph. If $f(v^*) \nmid d(v^*)$ for all $v^* \in V^*$, then G is of C_f 1.*

Proof. By Corollary 1.2, the following holds for every graph G

$$\Delta_f \le \chi'_f(G) \le \max_{v \in V} \left\{ \left\lceil \frac{d(v)+1}{f(v)} \right\rceil \right\}.$$

As $\Delta_f(G) = \max_{v \in V}\{\lceil \frac{d(v)}{f(v)} \rceil\}$, for each vertex $v \in V \setminus V^*$,

$$\left\lceil \frac{d(v)}{f(v)} \right\rceil < \Delta_f.$$

So

$$\left\lceil \frac{d(v)+1}{f(v)} \right\rceil \le \Delta_f.$$

On the other hand, for each vertex $v^* \in V^*$, $f(v^*) \nmid d(v^*)$ implies

$$\frac{d(v^*)}{f(v^*)} < \left\lceil \frac{d(v^*)}{f(v^*)} \right\rceil = \Delta_f.$$

Then

$$\frac{d(v^*)+1}{f(v^*)} \le \left\lceil \frac{d(v^*)}{f(v^*)} \right\rceil = \Delta_f, \qquad \left\lceil \frac{d(v^*)+1}{f(v^*)} \right\rceil = \Delta_f.$$

In all cases,

$$\max_{v \in V} \left\{ \left\lceil \frac{d(v)+1}{f(v)} \right\rceil \right\} \le \Delta_f.$$

Thus

$$\chi'_f(G) = \Delta_f.$$

This completes the proof. $\square$

Now we just concentrate on the cases where there exists a vertex $v^* \in V^*$ such that $f(v^*) | d(v^*)$. Let

$$V_0^* = \left\{ v : \Delta_f = \frac{d(v)}{f(v)}, v \in V \right\},$$
$$V_0 = \{v : \Delta_f \mid d(v), v \in V\}.$$

Obviously, $V_0^* \subseteq V^*$, $V_0^* = V^* \cap V_0$.

Given an edge-coloring of G with k colors $c_1, \ldots, c_k$, for $v \in V$, let $c_i(v)$ denote the set of edges incident to v with color c_i. Call an edge-coloring of G with k colors, $c_1, \ldots, c_k$, equitable if, for each $v \in V$,

$$||c_i(v)| - |c_j(v)|| \le 1 \quad (1 \le i < j \le k).$$

Let the *k-core* of G be the subgraph of G induced by the vertices v of G such that $k|d(v)$. The following lemma is given by Hilton and de Werra [5].

Lemma 2.2. *Let G be a graph and let $k \geq 2$. If the k-core of G is a set of isolated vertices, then G has an equitable edge-coloring with k colors.*

Theorem 2.3. *If the Δ_f-core of a graph G is a set of isolated vertices, then G is of C_f 1.*

Proof. If $\Delta_f = 1$, then $d(v) \leq f(v)$ for all $v \in V$. Obviously, we can f-color graph G with one color and thus G is of C_f 1. Assume that $\Delta_f \geq 2$. By Lemma 2.2, G has an equitable edge-coloring with Δ_f colors. We have known that $\Delta_f \geq \lceil \frac{d(v)}{f(v)} \rceil$ for each vertex $v \in V$. Therefore, $f(v) \geq \lceil \frac{d(v)}{\Delta_f} \rceil$. By the definition of equitable edge-coloring, we have $|c_i(v)| \leq \lceil \frac{d(v)}{\Delta_f} \rceil$ for each $i \in \{1, 2, \ldots, \Delta_f\}$ and each $v \in V$. That is to say, $f(v) \geq |c_i(v)|$ for each $i \in \{1, 2, \ldots, \Delta_f\}$ and each $v \in V$. Thus the equitable Δ_f edge-coloring is an f-coloring. By Corollary 1.2, $\chi'_f(G) = \Delta_f$. Therefore, G is of C_f 1. This completes the proof of the theorem. $\square$

We will extend Theorem 2.3 to a more general situation. Before that, we need some preliminary concepts from [6].

We denote by C the set of Δ_f colors used to f-color a graph G. An edge colored with color $c \in C$ is called a *c-edge*. Denote by $d(v, c)$ the number of c-edges of G incident with the vertex v, and define $m(v, c) = f(v) - d(v, c)$. G is *f-colored* if and only if $m(v, c) \geq 0$ for every $v \in V$ and $c \in C$. Color c is *available* at vertex v if $m(v, c) \geq 1$. Define $M(v) = \{c : m(v, c) \geq 1, c \in C\}$. A *walk* W is a sequence of distinct edges $v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$, where the vertices $v_0, v_1, \ldots, v_k$ are not necessarily distinct. Walk W is often denoted simply by $v_0v_1 \ldots v_k$. We call v_0 the *start vertex* of W and v_k the *end vertex*. The *length* of W is the number k of edges in W, and denoted by $|W|$. If $v_0 = v_k$, then W is called a *closed walk*. For two distinct colors $a, b \in C$, a walk W of length one or more is called an *ab-alternating walk* if W satisfies the following conditions:

(a) The edges of W are colored alternately with a and b with the first edge $e_1 = v_0v_1$ being colored b;
(b) $m(v_0, a) \geq 1$ if $v_0 \neq v_k$,
 $m(v_0, a) \geq 2$ if $v_0 = v_k$ and $|W|$ is odd;
(c) $m(v_k, b) \geq 1$ if $v_0 \neq v_k$ and $|W|$ is even,
 $m(v_k, a) \geq 1$ if $v_0 \neq v_k$ and $|W|$ is odd.

Note that any closed walk W of even length whose edges are colored with a and b alternately is an ab-alternating walk. If G is f-colored and W is an ab-alternating walk, then interchanging the colors a and b of the edges in walk W preserves an f-coloring of G. This operation is called *switching W*. When W is switched, $m(v_i, a)$ and $m(v_i, b)$ remain as they were if $i \neq 0, k$, while $m(v_0, b) \geq 1$ if W is not a closed walk of even length. Obviously, W is switched into a ba-alternating walk. We denote by $W(a, b; v_0)$ an ab-alternating walk which starts with vertex v_0. We write $d(v, c; G)$, $m(v, c; G)$ and $M(v; G)$ for $d(v, c)$, $m(v, c)$ and $M(v)$ of graph G respectively if confusions may occur.

"Shifting a fan" is another standard technique of the classical edge-coloring [7]. We also apply it in f-coloring. Let $e_0 = wv_0$ be an uncolored edge. Then a *fan* F is a sequence of distinct edges $e_0, e_1, \ldots, e_k$ incident with the vertex w such that there is a sequence of distinct colors $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ satisfying the following conditions (a)–(c), where w is called the *pivot* of F and v_i, $0 \leq i \leq k$, is the end of e_i other than w.

(a) α_i is an available color at v_i, $0 \leq i \leq k-1$;
(b) e_i, $1 \leq i \leq k$, is colored with α_{i-1}; and
(c) $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ are distinct.

Since G is a graph, vertices $v_0, v_1, \ldots, v_k$ are distinct. *Shifting a fan F* means to recolor e_i with α_i for each i, $0 \leq i \leq k-1$, and erase the color α_{k-1} of e_k. Shifting F yields another f-coloring of G in which e_k instead of e_0 is uncolored.

We next present another sufficient condition for a graph to be of C_f 1. It is more general than Theorem 2.3. Note that any subgraph G' of G has $f_{G'}(v) = f_G(v)$ for all $v \in V(G')$.

Theorem 2.4. *Let G be a graph and let G_0^* be the subgraph of G induced by the vertices of V_0^*. Then G is of C_f 1 if G_0^* is a forest.*

Proof. We complete this proof by coloring the edges of G with the $\Delta_f(G)$ colors of C in three steps. Let G_0 be the subgraph of G induced by V_0.

$\mathbf{1^0}$. Let $G^1 = G - G_0$, where $V(G^1) = V(G)$ and $E(G^1) = E(G) \setminus E(G_0)$. By the definition of f-coloring and $f_{G^1}(v) = f_G(v)$ for all $v \in V(G^1)$, it is easy to verify that $\Delta_f(G^1) \leq \Delta_f(G)$. If $\Delta_f(G^1) = \Delta_f(G)$, by Theorem 2.3, we have $\chi'_f(G^1) = \Delta_f(G^1) = \Delta_f(G)$. Otherwise, by Corollary 1.2, we have $\chi'_f(G^1) \leq \Delta_f(G^1) + 1 \leq \Delta_f(G)$. In any case, we can f-color G^1 with $\Delta_f(G)$ colors in C.

$\mathbf{2^0}$. Next we will color the edges in $E_1 = E(G_0) \setminus E(G_0^*)$. Let $G^2 = G - G_0^*$. For each edge $e = uv \in E_1$, it is easy to see that $M(u; G) \neq \phi$ and $M(v; G) \neq \phi$. It is sufficient to show that if $G^2 - e_0$ is edge-colored with $\Delta_f(G)$ colors, then G^2 can also be colored with $\Delta_f(G)$ colors. Assume that $G^2 - e_0$ is f-colored with $\Delta_f(G)$ colors in C and $e_0 \in E_1$. For the uncolored edge $e_0 = wv_0$, we construct a fan F as follows. Clearly the single edge e_0 is a fan. Assume in general that a fan $F = e_0, e_1, \ldots, e_k$ has been constructed so far and it has a color sequence $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$. If $M(w; G) \cap M(v_k; G) \neq \phi$, then there exists a color $\alpha_k \in M(w; G) \cap M(v_k; G)$ such that α_k is available at vertices w and v_k. Shifting the fan F and subsequently coloring e_k with α_k would complete an f-coloring of G^2 with $\Delta_f(G)$ colors. Thus one can assume that $M(w; G) \cap M(v_k; G) = \phi$. Furthermore, assume that F is a maximal fan. Let $\beta \in M(w; G)$ and $\alpha_k \in M(v_k; G)$. Since F is maximal, then it does not have a β-edge and has exactly one $\alpha_k (= \alpha_j)$-edge $e_{j+1} (0 \leq j \leq k-1)$. Note that $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ are all distinct. Thus any $\alpha_k\beta$-alternating walk $P = W(\alpha_k, \beta; v_k)$ of G passes through none of the edges of F except for the α_k-edge e_{j+1}. We consider four cases depending on the end vertex of P.

Case 1. If P ends at $v \in V(G) \setminus \{w, v_j, v_k\}$, then switch P and subsequently shift $F = e_0, e_1, \ldots, e_k$. Since β becomes an available color at both ends of the uncolored edge $e_k = wv_k$, coloring e_k with β would complete an f-coloring of G^2 with no more than $\Delta_f(G)$ colors.

Case 2. If P ends at w, then switch P and subsequently shift the subfan $F' = e_0, e_1, \ldots, e_j$. Since $\alpha_k = \alpha_j$ becomes an available color at both ends of the uncolored edge $e_j = wv_j$, coloring e_j with α_k would complete an f-coloring of G^2 with no more than $\Delta_f(G)$ colors.

Case 3. If P ends at v_j, then switch P and subsequently shift the subfan $F' = e_0, e_1, \ldots, e_j$. Since β becomes an available color at both ends of the uncolored edge $e_j = wv_j$, coloring e_j with β would complete an f-coloring of G^2 with no more than $\Delta_f(G)$ colors.

Case 4. If P ends at v_k, then P is an $\alpha_k\beta$-alternating closed walk and $|P|$ is odd because of $\beta \notin M(v_k; G)$. Furthermore, $m(v_k, \alpha_k; G) \geq 2$. Switching P and subsequently shifting the fan

$F = e_0, e_1, \ldots, e_k$ make β an available color at both ends of the uncolored edge $e_k = wv_k$. Coloring e_k with β would complete an f-coloring of G^2 with no more than $\Delta_f(G)$ colors.

3⁰. Finally, we color the edges in $E(G_0^*)$ with the colors in C. Denote by S the colored edge set of $E(G_0^*)$ and by H the vertex set of S. We color edges of G_0^* in the following way.

Step 0. $S = \phi$, $H = \phi$.
Step 1. Repeat until $S = E(G_0^*)$.
Substep 1. If, for each $e = uv \in E(G_0^*) \setminus S$, we have $u \notin H$ and $v \notin H$, then get an arbitrary edge $e_0 = wv_0 \in E(G_0^*) \setminus S$. Otherwise go to Substep 3.
Substep 2. Construct a fan $F = e_0, e_1, \ldots, e_k$ with w as the pivot. Color e_0 according to the method used in **2⁰**. Let $S = S \cup \{e_0\}$, $H = H \cup \{w, v_0\}$. If $S \neq E(G_0^*)$, go to Substep 1.
Substep 3. Select $e_0 = wv_0 \in E(G_0^*) \setminus S$ such that $w \notin H$ and $v_0 \in H$. Go to Substep 2.

According to the above coloring method, for each $e = wv_0 \in E(G_0^*) \setminus S$, we can always find a vertex $w \notin H$ as the pivot of F since G_0^* is acyclic. Obviously, $M(u; G) \neq \phi$ and $M(v; G) \neq \phi$ for each $e = uv \in E(G_0^*) \setminus S$. Furthermore, for each $u \in H$, if there exists an edge $e = uv \in E(G_0^*) \setminus S$, then $M(u; G) \neq \phi$; otherwise $M(u; G) = \phi$. For these reasons, it is impossible that there appears the edge $e_k = wv_k$ with $M(v_k; G) = \phi (k \geq 0)$ in $E(F)$, where w is the pivot of F. That is to say, we can always construct a fan $F = e_0, e_1, \ldots, e_k (k \geq 0)$ for each $e_0 \in E(G_0^*) \setminus S$ and eventually obtain an f-coloring of G using $\Delta_f(G)$ colors.

This completes the proof. □

If $f(v) = 1$ for all $v \in V(G)$, then $V_0^* = \{v : d(v) = \Delta(G), v \in V\}$. Denote by G_Δ the subgraph induced by the vertices of degree Δ in G. Thus we have the following corollary which is proved by Fournier [8].

Corollary 2.5. *If G is a graph such that G_Δ is a forest, then G is class 1.*

3. Some problems for future research

It is not difficult to see that the f-coloring problem has close relations with the equitable edge-coloring problem or the factorization problem. From the above two respects, we probably find some general methods of classification of graphs on f-colorings.

Finally we present the following problems for future research:

Problem 1. What are the necessary and sufficient conditions for a graph to be of C_f 1 or C_f 2?

Problem 2. Are the cases where $f(v) \geq 2$ for all $v \in V$ easier?

Problem 3. Classify some special classes of graphs.

Acknowledgment

The authors thank the referees for their valuable suggestions and for carefully reading the manuscript.

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